1 Dynamic programming: An example

Here we see how to solve an infinite Ramsey model with full depreciation $\delta = 1$ using backwards induction. This method has to do with iterating the value function up to a point where we are able to see the long term properties of the decision rule and the value function itself.

1.1 Problem set up

We set up an economy where one-infinitely living individual tries to maximize his consumption stream. Mathematically we have:

$$\underset{\{c_t,k_{t+1}\}_{t=0}^{\infty}}{\text{Max}} \qquad U = \sum_{t=0}^{\infty} \beta^t \ln c_t \tag{1}$$

subject to

$$c_t = Ak_t^{\alpha} - k_{t+1} \tag{2}$$

where k_0, $A \geq 0$, $\alpha \in (0,1)$ and $\beta \in (0,1)$.

1.2 Value Function Iteration

The value function for a random period t is

$$V_t = \underset{\{i_t\}}{\text{Max}} \qquad \ln(Ak_t^{\alpha} - i_t) + \beta V_{t+1} \tag{3}$$

Now we change the horizon. We set a period T as if it were the last period this individual will face and we start solving problem from T back.

1.2.1 Period T

For period T the Bellman equation in (3) becomes:

$$V_T = \underset{\{i_T\}}{\text{Max}} \qquad \ln(Ak_T^{\alpha} - i_T) + \beta V_{T+1} \tag{4}$$

and here we have the first problem. We do not know the value of V_{T+1}. But we know the invidual values nothing any consumption beyond the end of his life, therefore $k_{T+1} = 0 = i_T$ and $V_{T+1} = 0$. If we maximize[1] (4) we find that the value at T is

$$V_T = \ln A + \ln k_T^{\alpha} \tag{5}$$

and the decision rule tells the individual not to invest at all

$$i_T = 0 \tag{6}$$

1.2.2 Period T-1

We start solving for period $T-1$ using the same procedure. Let us write the Bellman equation for this period.

$$V_{T-1} = \underset{\{i_{T-1}\}}{\text{Max}} \qquad \ln(Ak_{T-1}^{\alpha} - i_{T-1}) + \beta V_T$$

but now we do know the value of V_T. If we plug it in the above equation we get closer to something we can maximize

$$V_{T-1} = \underset{\{i_{T-1}\}}{\text{Max}} \qquad \ln(Ak_{T-1}^{\alpha} - i_{T-1}) + \beta \ln A + \beta \ln k_T^{\alpha}$$

Still we have an unknown term, k_T. But we can use the law of motion that says $k_T = i_{T-1}$. Let us put this piece of information in the value function to maximize it:

$$V_{T-1} = \underset{\{i_{T-1}\}}{\text{Max}} \qquad \ln(Ak_{T-1}^{\alpha} - i_{T-1}) + \beta \ln A + \beta \ln i_{T-1}^{\alpha} \tag{7}$$

[1] Actually we cannot maximize a constant, it is the value itself what we get

and the first order neccesary condition becomes

$$\frac{\partial U}{\partial i_{T-1}} = -\frac{1}{Ak_{T-1}^{\alpha} - i_{T-1}} + \frac{\alpha\beta}{i_{T-1}} = 0$$

Rearrange this equation so as to obtain an expression with i_{T-1} as a function of current capital and other parameters like

$$i_{T-1} = \frac{\alpha\beta A k_{T-1}^{\alpha}}{1+\alpha\beta} \tag{8}$$

Finally we need the solution for the value at $T-1$. We have to plug (8) into (7) to obtain

$$V_{T-1} = \ln\left(Ak_{T-1}^{\alpha} - \frac{\alpha\beta A k_{T-1}^{\alpha}}{1+\alpha\beta}\right) + \beta\ln A + \beta\ln\left(\frac{\alpha\beta A k_{T-1}^{\alpha}}{1+\beta\alpha}\right)^{\alpha}$$

develop the neperian logs and simplify for k_{T-1}

$$V_{T-1} = \ln\frac{A}{1+\alpha\beta} + \beta\ln A + \alpha\beta\ln\frac{\alpha\beta A}{1+\alpha\beta} + \alpha(1+\alpha\beta)k_{T-1} \tag{9}$$

We are ready to proceed to the next step but although the above equation is what we need to assess the trend of the value function at the end of the example it is not very handy to work with. We will use a simpler equation grouping all the constant terms together

$$V_{T-1} = C + D\ln k_{T-1}$$

1.2.3 Period T-2

Here we follow the steps we undertook when we solved for $T-1$. Let us start with the Bellman equation for $T-2$.

$$V_{T-2} = \underset{\{i_{T-2}\}}{\text{Max}} \quad \ln(Ak_{T-2}^{\alpha} - i_{T-2}) + \beta V_{T-1}$$

again we know V_{T-1} so let us use it -the short one of course-.

$$V_{T-2} = \underset{\{i_{T-2}\}}{\text{Max}} \quad \ln(Ak_{T-2}^{\alpha} - i_{T-2}) + \beta C + \beta D\ln k_{T-1}$$

and make use of the law of motion $k_{T-1} = i_{T-2}$

$$V_{T-2} = \underset{\{i_{T-2}\}}{\text{Max}} \quad \ln(Ak_{T-2}^{\alpha} - i_{T-2}) + \beta C + \beta D\ln i_{T-2} \tag{10}$$

the first partial derivative with respect to investment yields:

$$\frac{\partial U}{\partial i_{T-2}} = -\frac{1}{Ak_{T-2}^{\alpha} - i_{T-2}} + \frac{\beta D}{i_{T-2}} = 0$$

the solution depends on D

$$i_{T-2} = \frac{\beta D A k_{T-2}^{\alpha}}{1+\beta D}$$

if we substitute for D we have the decision rule:

$$i_{T-2} = \frac{(\beta\alpha + (\beta\alpha)^2)Ak_{T-2}^{\alpha}}{1+\beta\alpha+(\beta\alpha)^2} \tag{11}$$

Now we need the value of the Bellman equation at $T-2$. Plug the decision rule that contains D in equation (10).

$$V_{T-2} = \ln\left(Ak_{T-2}^{\alpha} - \frac{\beta D A k_{T-2}^{\alpha}}{1+\beta D}\right) + \beta C + \beta D\ln\left(\frac{\beta D A k_{T-2}^{\alpha}}{1+\beta D}\right)$$

Now we will obtain two equations. A value as a function of the model parameters and current capital and another value as a function of two new black boxes E and F. The latter can be done from the above equation using some properties of the logs:

$$V_{T-2} = \ln\frac{A}{1+\beta D} + \beta C + \beta D\ln\frac{\beta D A}{1-\beta D} + \alpha(1+\beta D)\ln k_{T-2}$$

group all the constant terms altogether and obtain the equation we need to keep iterating

$$V_{T-2} = E + Fk_{T-2}$$

and the complete equation, after a sizeable amount of algebra becomes (be patient if you want to work this out)

$$V_{T-2} = \ln \frac{A}{1+\alpha\beta+(\alpha\beta)^2} + \beta \ln \frac{A}{1+\alpha\beta} + \beta^2 \ln A + \alpha\beta^2 \ln \frac{\alpha\beta A}{1+\alpha\beta} + \alpha\beta(1+\alpha\beta) \ln \frac{\beta\alpha(1+\alpha\beta)A}{1+\alpha\beta+(\alpha\beta)^2} + \alpha(1+\alpha\beta+(\alpha\beta)^2) \ln k_{T-2} \tag{12}$$

If we look at this equation we will be able to find 4 patterns. The first is made of the first 3 terms, the second one is the fourth term, the third comes from the fifth term and finally we have the recursive equation with the capital. Actually what we have is enough to capture the long-run properties of the model but you may not neccesarily know this so let us check there is no new pattern in the next iteration before solving for the limit.

1.2.4 Period T-3

First things first, the Bellman equation for $T-3$:

$$V_{T-3} = \underset{\{i_{T-3}\}}{\text{Max}} \quad \ln(Ak_{T-3}^{\alpha} - i_{T-3}) + \beta V_{T-2}$$

Now we use V_{T-2}, again, the short one, to obtain:

$$V_{T-3} = \underset{\{i_{T-3}\}}{\text{Max}} \quad \ln(Ak_{T-3}^{\alpha} - i_{T-3}) + \beta E + \beta F k_{T-2}$$

and finally let us make use of the law of motion, $k_{T-2} = i_{T-3}$

$$V_{T-3} = \underset{\{i_{T-3}\}}{\text{Max}} \quad \ln(Ak_{T-3}^{\alpha} - i_{T-3}) + \beta E + \beta F i_{T-3} \tag{13}$$

Derive with respect the control variable and we have

$$\frac{\partial U}{\partial i_{T-3}} = -\frac{1}{Ak_{T-3}^{\alpha} - i_{T-3}} + \frac{\beta F}{i_{T-3}} = 0$$

Solve for i_{T-3}

$$i_{T-3} = \frac{\beta F A k_{T-3}^{\alpha}}{1+\beta F}$$

Now plug F in the above equation to obtain the decision rule at period $T-3$

$$i_{T-3} = \frac{(\alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3) A k_{T-3}^{\alpha}}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3} \tag{14}$$

We will be no longer iterating so we just need the real expression of the value function at $T-3$. In order to obtain so we have to plug C,D,E and F in V_{T-3}[2]. After tons of careful rearrangements and some algebra we obtain the following expression:

$$\begin{aligned} V_{T-3} = {} & \ln\left(\frac{A}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3}\right) + \beta \ln\left(\frac{A}{1+\alpha\beta+(\alpha\beta)^2}\right) + \beta^2 \ln \frac{A}{1+\beta\alpha} + \beta^3 \ln A \\ & + \beta^2\alpha\beta \ln \frac{\alpha\beta A}{1+\alpha\beta} + \beta\alpha\beta \ln \frac{(\alpha\beta+(\alpha\beta)^2)A}{1+\alpha\beta+(\alpha\beta)^2} \\ & + (\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3) \ln \frac{A(\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3)}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3} \\ & + (1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3) \ln k_{T-3}^{\alpha} \end{aligned} \tag{15}$$

[2] Actually there is no C in V_{T-3} but it is inside E.

1.3 Arbitrary t and limit solution

We are almost done. We just need to take a look at the two things we are interested in; the decision rules and the value functions we have obtained in the last section. Let us start with the former. Equations 6,8,11 and 14 contain the decision rules we have obtained so far. Following the pattern we can obtain the closed-loop solution for an arbitrary t as:

$$k_{T-t+1} = i_{T-t} = \frac{Ak^{\alpha}_{T-t}\sum_{i=1}^{t}(\alpha\beta)^i}{1+\sum_{i=1}^{t}(\alpha\beta)^i} \tag{16}$$

and the solution when t tends to infinite becomes:

$$\lim_{\{t\to\infty\}} \quad k_{T-t+1} = \lim_{\{t\to\infty\}} \quad i_{T-t} = \alpha\beta A k^{\alpha}_{T-t} \tag{17}$$

Now it is time for the value function. Remember when building this function that there are 4 recursives equation inside the value function, or in other words, we have to work out 4 expressions with a sum. For an arbitrary t the value function looks as follows:

$$V_{T-t} = \sum_{i=0}^{t}\beta^i \ln\left(\frac{A}{\sum_{i=0}^{t}(\alpha\beta)^i}\right) + \sum_{i=0}^{t}\beta^{i+1}\alpha\beta\ln\left(\frac{A\sum_{i=0}^{t}(\alpha\beta)^{i+1}}{\sum_{i=0}^{t}(\alpha\beta)^i}\right) + \tag{18}$$
$$\sum_{i=0}^{t}(\alpha\beta)^{i+1}\left(\ln\left(\frac{A\sum_{i=0}^{t}(\alpha\beta)^{i+1}}{\sum_{i=0}^{t}(\alpha\beta)^i}\right)\right) + \ln k^{\alpha}_{T-t}\sum_{i=0}^{t}(\alpha\beta)^i \quad \text{for all } t\geq 2$$

Finally we have to apply the geometric series formula repeatedly so as to obtain the value when t tends to infinity.

$$\lim_{\{t\to\infty\}} \quad V_{T-t} = \frac{\ln A(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln(A\alpha\beta)}{1-\beta} + \frac{\ln k^{\alpha}}{1-\alpha\beta} \tag{19}$$

where I have merged the second and the third expressions in (18) after simplifying.